Well posed constraint-preserving boundary conditions
for the linearized Einstein equations

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In the Cauchy problem of general relativity one considers initial data that satisfies certain constraints. The evolution equations guarantee that the evolved variables will satisfy the constraints at later instants of time. This is only true within the domain of dependence of the initial data. If one wishes to consider situations where the evolutions are studied for longer intervals than the size of the domain of dependence, as is usually the case in three dimensional numerical relativity, one needs to give boundary data. The boundary data should be specified in such a way that the constraints are satisfied everywhere, at all times. In this paper we address this problem for the case of general relativity linearized around Minkowski space using the generalized Einstein-Christoffel symmetric hyperbolic system of evolution equations. We study the evolution equations for the constraints, specify boundary conditions for them that make them well posed and further choose these boundary conditions in such a way that the evolution equations for the metric variables are also well posed. We also consider the case of a manifold with a non-smooth boundary, as is the usual case of the cubic boxes commonly used in numerical relativity. The techniques discussed should be applicable to more general cases, as linearizations around more complicated backgrounds, and may be used to establish well posedness in the full non-linear case.

I. INTRODUCTION

In the Cauchy problem of general relativity, Einstein's equations split into a set of evolution equations and a set of constraint equations. Given initial data that satisfies the constraints one can show that the evolution equations imply that the constraints continue to hold in the domain of dependence of the initial slice. However, in numerical relativity, due to finite computer resources, one usually solves the field equations in a domain of the form $(t, x^i) \in \mathcal{R} \times \Omega$, with Ω a bounded space-like surface with boundary $\partial\Omega$. In this case the domain of dependence of the initial slice is "too small" and one wants to evolve beyond it. Then the question that arises is what boundary conditions to give at the artificial boundary $\mathcal{R} \times \partial\Omega$. This problem has two aspects to it. First, one wishes to prescribe boundary conditions that make the problem a well posed one which preserves the constraints. In addition to that, one might desire to embed in the boundary conditions some physically appealing property (for instance that no gravitational radiation enters the domain). These two aspects are in principle separate. In this paper we will concentrate on the first one, namely, how to prescribe consistent boundary conditions. The construction we propose ends up giving "Dirichlet or Neumann-like" boundary conditions on some components of the metric, with some free sources. The latter allow freedom to consider boundary conditions for any spacetime in any slicing.

If the field equations are cast into a first order symmetric hyperbolic form there is a known prescription for achieving well posedness for an initial-boundary value problem (this is discussed in detail in section II), namely maximal dissipative boundary conditions [1]. Well posedness is a necessary condition for implementing a stable numerical code in the sense of Lax's theorem [2]. However, additional work is needed if one wishes to have a code that is not only stable but that preserves the constraints throughout the domain of evolution. As we stated above, this implies giving initial and boundary data that guarantees that the constraints are satisfied. Traditionally, most numerical relativity treatments have been careful to impose initial data that satisfies the constraints. However, very rarely boundary conditions that lead to well posedness are used, and much less frequently are they consistent with the constraints. Only recently, following Friedrich and Nagy [3], has work with numerical relativity in mind started along these lines [4],[5],[6],[7], [8]. Of particular interest is the recent paper of Szilagyi and Winicour [9], which outlines a construction with several points in common with the one we describe here.

In this article we derive well posed, constraint-preserving boundary conditions for the generalized [10] Einstein-Christoffel (EC) [11] symmetric hyperbolic formulations of Einstein's equations, when linearized around flat spacetime. The procedure consists in studying the evolution system for the constraints and making sure that the corresponding initial-boundary value problem is well-posed through appropriate boundary conditions. There is some freedom in the specification of the latter. Next, we translate them into boundary conditions for the variables of the main evolution

system. The main difficulty consists in making use of the freedom that one has in the boundary conditions for the evolution system of the constraints in such a way that the resulting boundary conditions for the main evolution system ensure well-posedness. We consider the case of a non-smooth cubic boundary as is of interest in numerical relativity. This involves the additional complication of being careful about ensuring compatibility at the edges joining the faces. To our knowledge, this is the first detailed analysis of the initial-boundary value problem with a non-smooth boundary in the gravitational context.

The organization of this paper is as follows: in section II we review the basic technique of energy estimates used to prove well posedness. In section III we review the generalized EC symmetric hyperbolic formulation, write down the evolution system for the constraints and analyze under what conditions the latter is symmetric hyperbolic. In section IV we compute the characteristic variables for the main evolution system and for the system that evolves the constraints. These characteristic variables are a key element for prescribing the boundary conditions that lead to well-posedness. In section V we write down the constraint preserving boundary conditions in terms of the variables of the main evolution system. In section VI we derive the necessary energy estimates to show that all the systems involved in our constraint-preserving treatment are well posed. In section VII we summarize the constraint-preserving construction of this paper. We end with a discussion of possible future improvements to and applications of the boundary conditions introduced in this paper.

II. BASIC ENERGY ESTIMATES

In this section we review some basic notions of energy estimates for systems of partial differential equations, as discussed, for instance, in [12]. Consider a first order in time and space linear evolution system of the form

$$\partial_t u = A^j \partial_j u, \tag{1}$$

where $u = u(t, x^i)$ is a vector valued function, $t \geq 0$, $x^i \in \Omega$, and where the matrices A^1, A^2 and A^3 are constant. The symbol $P(\vec{n})$ is defined as $P := A^i n_i$. The system is symmetric (or symmetrizable) hyperbolic if there is a symmetrizer for P. That is, a positive definite, Hermitian matrix H *independent of* n^i such that $HP = P^\dagger H$ for all n_i with $\sum_{i=1\ldots3} n_i^2 = 1$.

In order to show well-posedness of the initial value problem for such a system, one usually derives a bound for the energy

$$E(t) = \int_\Omega (u, Hu) \, d^3x , \tag{2}$$

where $(.,.)$ denotes the standard scalar product. Taking a time derivative of (2), using equation (1), the fact that HA^j are symmetric matrices, and Gauss' theorem, one obtains

$$\frac{d}{dt} E(t) = \int_\Omega 2(u, HA^j \partial_j u) \, d^3x = \int_\Omega \partial_j (u, HA^j u) \, d^3x = \int_{\partial\Omega} (u, HP(\vec{n})u) \, d\sigma,$$

where $\vec{n}$ is now the unit outward normal to the boundary $\partial\Omega$ of the domain. For simplicity let's assume that $P(\vec{n})$ has only the eigenvalues ± 1 and 0. Let $u^{(+)}$, $u^{(-)}$, and $u^{(0)}$ denote the projections of u onto the eigenspaces corresponding to the eigenvalues 1, -1 and 0, respectively. That is, $u = u^{(+)} + u^{(-)} + u^{(0)}$ and $P(\vec{n})u = u^{(+)} - u^{(-)}$. Then

$$(u, HP(\vec{n})u) = (u^{(+)}, Hu^{(+)}) - (u^{(-)}, Hu^{(-)}).$$

If we impose boundary conditions of the form $u^{(+)} = Ru^{(-)}$ with R "small enough" so that $R^T HR \leq H$ [19], it follows that $E(t) \leq E(0)$ for all $t \geq 0$. These kind of energy estimates are a key ingredient in well-posedness proofs. One can generalize this result to boundary conditions of the form

$$u^{(+)} = Ru^{(-)} + g, \tag{3}$$

where $g = g(t, x^A)$ is a prescribed function at the boundary and $R^T HR \leq H$, as before. In this case, an energy estimate can be obtained as follows: We first choose a function ψ that satisfies $\psi^{(+)} = R\psi^{(-)} + g$ at the boundary. Then, we consider the variable $\tilde{u} \equiv u - \psi$ instead of u which now satisfies the homogeneous boundary condition $\tilde{u}^{(+)} = R\tilde{u}^{(-)}$ and the modified evolution equation

$$\partial_t \tilde{u} = A^j \partial_j \tilde{u} + F,$$

where $F = A^j \partial_j \psi - \partial_t \psi$ is a forcing term. One now repeats the estimate for the energy defined in (2), with u replaced by $\tilde{u}$, and obtains

$$\frac{d}{dt} E(t) \leq 2 \int_\Omega (\tilde{u}, HF) d^3 x \leq 2\|\tilde{u}\| \cdot \|F\|,$$

where $\|\tilde{u}\| \equiv E^{1/2}$, $\|F\| \equiv (\int (F, HF) d^3 x)^{1/2}$ and where we have used Schwarz's inequality. Therefore, we obtain the estimate

$$\|\tilde{u}(t, .)\| \leq \|\tilde{u}(0, .)\| + \int_0^t \|F(s, .)\| ds.$$

Similar estimates can be obtained for systems with non-constant coefficients, as is the case, for instance, of linearizations around a given non-Minkowski metric, as in black hole perturbations. In the non-linear case the proofs of bounds are only for finite amounts of time, since solutions can blow up in a finite time starting from initial data with finite energy.

The existence of an energy estimate implies that the initial-boundary value problem is well posed. By this we mean that there exists a unique smooth solution to the problem for which the energy estimate holds [13, 14]. It should be noted that these results are in principle only valid for smooth boundaries. Later in this paper we will discuss boundaries that are non-smooth. It should be understood that in those cases the energy estimates we derive do not necessarily imply the existence and uniqueness of smooth solutions.

III. THE FIELD EQUATIONS AND THEIR CONSTRAINT PROPAGATION

In this section we present the field equations that we will use through this paper, and analyze the constraints propagation. In the first subsection we present the evolution equations for the main variables within the generalized Einstein-Christoffel symmetric hyperbolic formulation of Einstein's equations. In the second subsection we analyze, in the fully non-linear case, the evolution equations for the constraints within this formulation and derive necessary conditions for the latter to be symmetrizable. We will need this system to be symmetric hyperbolic in order to derive an energy estimate in the way we sketched in Section II. Rather surprisingly, it turns out that in the original EC system the constraints' propagation does not seem to be symmetrizable. Imposing the symmetric hyperbolicity condition naturally restricts the free parameter of the system to an open interval. As a side note, there seems to be some correlation between the stability properties of the system found in numerical experiments and this natural choice.

A. The formulation

In [10] the following symmetric hyperbolic system [20] of evolution equations for the three-metric (g_{ij}), the extrinsic curvature (K_{ij}), and some extra variables f_{kij} that are introduced in order to make the system first order in space is derived:

$$\partial_0 g_{ij} = -2K_{ij}, \tag{4}$$
$$\partial_0 K_{ij} = -\partial^k f_{kij} + l.o., \tag{5}$$
$$\partial_0 f_{kij} = -\partial_k K_{ij} + l.o., \tag{6}$$

where $\partial^k \equiv g^{kl} \partial_l$ and where $\partial_0 = (\partial_t - \mathcal{L}_\beta)/N$ is the derivative operator along the normal to the spatial $t = const.$ slices. The shift β^i is an a priori prescribed function on spacetime while the lapse N is determined by $N = g^{1/2} e^Q$, with Q a priori prescribed. Here and in the following $l.o.$ stands for "lower order terms". These terms depend on g_{ij}, K_{ij}, f_{kij}, Q and β^i but not on the derivatives of g_{ij}, K_{ij} or f_{kij}. Provided that the constraints (see equation (8) below) are satisfied, the spatial derivatives, $d_{kij} \equiv \partial_k g_{ij}$, of the three-metric are obtained from

$$d_{kij} = 2 f_{kij} + \eta \, g_{k(i} \left(f_{j)s}{}^s - f^s{}_{j)s} \right) + \frac{\eta - 4}{4} \, g_{ij} \left(f_{ks}{}^s - f^s{}_{ks} \right), \tag{7}$$

where η is a free parameter with the only restriction $\eta \neq 0$.

The evolution system (4,5,6), besides being symmetric hyperbolic, has the additional feature that all characteristic speeds (with respect to the normal derivative operator ∂_0) are either 0 or ± 1, so that all characteristic modes lie either along the light cone or along the orthogonal to the hypersurfaces $t = const.$ direction.

The particular case with $\eta = 4$ corresponds to the system derived in [11]. As we will show shortly, the latter does, however, not seem to admit a symmetric hyperbolic formulation for the evolution of the constraints.

B. The evolution of the constraints

In order to solve Einstein's vacuum equations, one has to supplement the evolution equations (4,5,6) with the following constraints:

$$\begin{aligned}
0 = & \ C \equiv \tfrac{1}{2} g^{ab} \partial^k (d_{abk} - d_{kab}) + l.o. \, , \\
0 = & \ C_j \equiv \partial^a K_{aj} - g^{ab} \partial_j K_{ab} + l.o. \, , \\
0 = & \ C_{kij} \equiv d_{kij} - \partial_k g_{ij} \, , \\
0 = & \ C_{lkij} \equiv \partial_{[l} d_{k]ij} \, .
\end{aligned} \tag{8}$$

where C, C_i are the Hamiltonian and momentum constraints, respectively.

Using the evolution system (4,5,6) for the main variables, one obtains the following principal part for the evolution system for the constraint variables [10]:

$$\partial_0 C = \frac{\eta}{4} \partial^k C_k + l.o. \, , \tag{9}$$

$$\partial_0 C_i = \frac{4 - 2\eta}{\eta} \partial_i C - \partial^k C^s_{kis} - \partial^k C_{kis}{}^s + l.o. \, , \tag{10}$$

$$\partial_0 C_{kij} = l.o. \, , \tag{11}$$

$$\partial_0 C_{lkij} = \eta \, \partial_{[l} g_{k](i} C_{j)} + \frac{\eta - 4}{4} g_{ij} \partial_{[l} C_{k]} + l.o. \, . \tag{12}$$

A system is symmetrizable if we can find a transformation that brings the principal part into symmetric form. In order to investigate under which conditions this system is symmetrizable hyperbolic, we split C_{lkij} in its trace and trace-less parts:

$$C_{lkij} = E_{lkij} + \frac{1}{2} \left(g_{l(i} B_{j)k} - g_{k(i} B_{j)l} \right) + \frac{1}{3} g_{ij} W_{lk} \, ,$$

where E_{lkij} is trace-less with respect to all pair of indices and where in terms of the traces $S_{ki} \equiv C^s_{(ki)s}$, $A_{ki} \equiv C^s_{[ki]s}$ and $V_{lk} \equiv C_{lks}{}^s$,

$$B_{ki} = \frac{4}{3} S_{ki} - \frac{12}{5} A_{ki} - \frac{4}{5} V_{ki} \, , \qquad W_{lk} = \frac{9}{5} V_{lk} + \frac{12}{5} A_{lk} \, .$$

Rewriting the principal part in terms of these variables, we find

$$\partial_0 C = \frac{\eta}{4} \partial^k C_k + l.o. \, , \tag{13}$$

$$\partial_0 C_i = \frac{4 - 2\eta}{\eta} \partial_i C - \partial^k S_{ki} - \partial^k A_{ki} - \partial^k V_{ki} + l.o. \, , \tag{14}$$

$$\partial_0 S_{ki} = -\frac{3\eta}{4} \left(\partial_{(k} C_{i)} - \frac{1}{3} g_{ki} \partial^s C_s \right) + l.o. \, , \tag{15}$$

$$\partial_0 A_{ki} = (1 - \eta) \, \partial_{[k} C_{i]} + l.o. \, , \tag{16}$$

$$\partial_0 V_{ki} = \left(\frac{7\eta}{4} - 3 \right) \partial_{[k} C_{i]} + l.o. \, , \tag{17}$$

$$\partial_0 C_{kij} = l.o. \, . \tag{18}$$

$$\partial_0 E_{lkij} = l.o. \, . \tag{19}$$

Having split all the variables into their trace and trace-less parts the only natural transformations that remain are rescaling of the variables or linear combinations of the antisymmetric symbols A_{ki} and V_{ki}. First, looking at the terms depending on C_i in the evolution equation for C and vice-versa, we see that $\eta < 2$ is needed in order to make the corresponding block in the principal part symmetric by a rescaling of C. Next, looking at the terms involving S_{ki} in the evolution equation for C_i and vice-versa, through a similar reasoning, we obtain the condition $\eta > 0$. Finally, if $0 < \eta < 2$, it is easy to see that the transformation

$$\tilde{A}_{ki} = A_{ki} + V_{ki} \, , \qquad \tilde{V}_{ki} = (7\eta/4 - 3) A_{ki} + (\eta - 1) V_{ki} \, ,$$

brings the corresponding block into manifestly symmetric form.

We can summarize the result as follows: If $0 < \eta < 2$, the principal part of the system (13-19) is symmetric with respect to the inner product associated with

$$(U, U) \equiv \frac{16 - 8\eta}{\eta^2} CC + C^i C_i + \frac{4}{3\eta} S^{ki} S_{ki} + \frac{4}{8 - 3\eta} \tilde{A}^{ki} \tilde{A}_{ki} + \tilde{V}^{ki} \tilde{V}_{ki} + C^{kij} C_{kij} + E^{lkij} E_{lkij} \,, \qquad (20)$$

where $U = (C, C_i, S_{ki}, \tilde{A}_{ki}, \tilde{V}_{ki}, E_{lkij})^T$. It is interesting to notice that in a numerical empirical search for a value of η that improves the stability of a single black hole evolution, Kidder, Scheel and Teukolsky found the value $\eta = 4/33$, which lies inside the interval $0 < \eta < 2$ [10]. On the other hand, the evolution of the original EC ($\eta = 4$), for which we were not able to find a symmetrizer, according to [10] does not perform very well in 3D black hole evolutions. Also of interest is that in the recent work of Lindblom and Scheel [15] they note that the previously mentioned range of η is also preferred. (See figure 5 of their paper; the range $0 < \eta < 2$ translates to $-\infty < \gamma < -0.5$). In that work they also study the dependence on another parameter $\hat{z}$, which corresponds to a rescaling of a variable and therefore its effects do not influence the principal part of the equations, which is what determines the level of hyperbolicity of the system.

IV. CHARACTERISTIC VARIABLES

Here we discuss the characteristic variables of the main evolution system (4,5,6) and of the evolution system (13-19) for the constraints. The characteristic variables are needed in order to give the boundary conditions of type (3), which yield a well-posed initial-boundary value problem.

From now on we will concentrate on linearized (around Minkowski spacetime in Cartesian coordinates) gravity for simplicity. That is, the background metric is

$$ds^2 = -dt^2 + \delta_{ij} dx^i dx^j \,,$$

where δ_{ij} is the flat space metric. We also assume that our perturbations have vanishing shift and vanishing linearized densitized lapse. In these cases, all lower order terms vanish (with the obvious exception of the right hand side (RHS) of (4)) and the evolution equations simplify considerably.

We also choose our spatial domain to be a box $\Omega = [x_{min}, x_{max}] \times [y_{min}, y_{max}] \times [z_{min}, z_{max}]$, though the analysis below could be generalized to other domains.

A. Characteristic variables for the main system

When linearized around flat spacetime, the main evolution system reduces to

$$\begin{aligned}
\partial_t g_{ij} &= -2K_{ij} \,, \\
\partial_t K_{ij} &= -\partial^k f_{kij} \,, \qquad (21) \\
\partial_t f_{kij} &= -\partial_k K_{ij} \,. \qquad (22)
\end{aligned}$$

Since g_{ij} does not appear in the evolution equations for K_{ij} and f_{kij}, we do not consider its evolution equation in the following. Therefore, the system we consider has the simple form $\partial_t u = A^j \partial_j u$, with $u = (K_{ij}, f_{kij})^T$.

The characteristic variables with respect to a direction n^i are variables with respect to which the symbol $P(\vec{n}) = A^j n_j$ is diagonal. They can be obtained by first finding a complete set, $e_1, ..., e_{24}$, of eigenvectors of the symbol (the e_j's are called characteristic modes) and then expanding u with respect to these vectors. The coefficients in this expansion are called the characteristic variables.

For the above system there are six characteristic variables with speed 1, six with speed -1 and twelve with zero speed. They are given by

$$\begin{aligned}
v_{ij}^{(+)} &= K_{ij} - f_{nij} \,, & \text{(speed +1)} \\
v_{ij}^{(-)} &= K_{ij} + f_{nij} \,, & \text{(speed -1)} \qquad (23) \\
v_{kij}^{(0)} &= f_{kij} - n_k f_{nij} \,, & \text{(speed 0)}
\end{aligned}$$

where $f_{nij} \equiv f_{kij} n^k$. In terms of these variables, the evolution system becomes

$$\partial_t v_{ij}^{(\pm)} = \pm \partial_n v_{ij}^{(\pm)} - \delta^{kl} \partial_k^T v_{lij}^{(0)} \,, \qquad (24)$$

$$\partial_t v_{kij}^{(0)} = -\frac{1}{2} \partial_k^T \left(v_{ij}^{(+)} + v_{ij}^{(-)} \right) \,, \qquad (25)$$

where ∂_n denotes the derivative in the direction of n and $\partial_k^T \equiv \partial_k - n_k \partial_n$ are the derivatives with respect to the directions that are orthogonal to n.

1. Gauge, physical, and constraint-violating modes

Before we proceed, it is interesting to give the following interpretation to the characteristic variables of the main system with respect to a fixed direction n^k: Consider a Fourier mode of u with wave-vector along n^k, i.e. assume that the spatial dependence of u has the form $e^{in_j x^j}$. In this case, the linearized constraints assume the form $L(n^i)u = 0$, where $L(n^i)$ is a constant matrix. Also, since in the case we are considering the non principal terms vanish, characteristic modes of this form solve the evolution equations.

Now we can check which characteristic modes (or combination thereof) satisfy the constraint equations and which do not. It turns out that all modes violate the constraints, except for the ones corresponding to the characteristic variables $v_{nn}^{(\pm)}$ and $\hat{v}_{AB}^{(\pm)} \equiv v_{AB}^{(\pm)} - \delta_{AB} v_C^{C\,(\pm)}/2$, where A, B denote directions which are orthogonal to n.

Next, consider an infinitesimal coordinate transformation $x^\mu \mapsto x^\mu + X^\mu$. With respect to it,

$$K_{ij} \;\mapsto\; K_{ij} - \partial_i \partial_j X^t \;,$$
$$g_{ij} \;\mapsto\; g_{ij} + 2\partial_{(i} X_{j)} \;.$$

Assuming that X^t and X_j are proportional to $e^{in_j x^j}$ and using (7) we find that

$$v_{nn}^{(\pm)} \mapsto v_{nn}^{(\pm)} + X^t \pm X_n \;, \qquad v_{Aij}^{(0)} \mapsto v_{Aij}^{(0)} - \delta_{A(i}\left(n_{j)} X_n - X_{j)}\right) + \frac{\eta - 4}{4\eta}\,\delta_{ij} X_A \;,$$

while the remaining variables are invariant. In particular, we can gauge away the variables $v_{nn}^{(+)}$ and $v_{nn}^{(-)}$. The characteristic variables $\hat{v}_{AB}^{(\pm)}$ are gauge-invariant and satisfy the constraints.

This suggests the following classification: We call

- the variables $v_{nn}^{(\pm)}$ *gauge variables*.

- the variables $\hat{v}_{AB}^{(\pm)}$ *physical variables*.

- and the remaining variables $v_B^{B\,(\pm)}$ and $v_{nB}^{(\pm)}$ *constraint violating variables*.

We stress that this classification is exact only if all the fields depend on the variable $n_k x^k$ and the time coordinate t only. (In particular, it is exact in $1+1$ dimensions since then there is only one space dimension.) Nevertheless, this classification sheds light on the boundary treatment below: For example, we will see that giving boundary data that is consistent with the constraints will fix a combination of the in- and outgoing constraint violating variables while we will be free to choose any data for some combination of the in- and outgoing gauge and physical variables (see equation (38) below).

B. Characteristic variables for the constraints

For our purpose, it is sufficient to find the constraint characteristic variables that have non-zero speed, since these are the ones that enter the boundary condition (3). In the system considered here there are three characteristic variables with speed 1 and three with speed -1,

$$V_i^{(\pm)} = -C_i \pm \frac{2\eta - 4}{\eta}\,C n_i \pm (S_{ni} + \tilde{A}_{ni}), \tag{26}$$

while the remaining variables have zero speed.

V. CONSTRAINT-PRESERVING BOUNDARY CONDITIONS

Here we start with boundary conditions for the constraints that ensure that they are preserved though evolution and then translate them into the boundary conditions for the main system. In order to do so we write the in- and

outgoing characteristic variables of the constraints in terms of (derivatives of) the characteristic variables of the main system. Well-posedness of the resulting constraint system is shown in the Section VI.

In order to preserve the constraints we impose boundary conditions for their evolution of the form (3) with $g = 0$ ($g \neq 0$ would not preserve the constraints). That is,

$$V_i^{(+)} = L_i{}^j V_j^{(-)}, \tag{27}$$

where the matrix L is constant. As discussed in Sect. II, the coupling matrix L must be "small enough" if we want to obtain a useful energy estimate. We will analyze this question in the Section VI

Next, our task is to translate the conditions (27) into conditions on the main variables. Using the definition of the constraints, Eqs. (8), we find

$$C = \frac{\eta}{4} \partial^k v_k , \tag{28}$$

$$C_i = \partial^j K_{ij} - \partial_i K , \tag{29}$$

$$S_{ni} + \tilde{A}_{ni} = \partial^l f_{nil} - \partial_i f^s{}_{sn} + \frac{\eta}{4} \left[n_i \partial^l v_l - 3 \partial_i v_n \right] , \tag{30}$$

where $v_k = f_{ks}{}^s - f^s{}_{sk}$. Using Eqs. (28,29,30) and the definition of the characteristic variables, Eqs. (23,26), one obtains

$$V_n^{(\pm)} = \partial_n v_{BB}^{(\pm)} - \partial^B v_{Bn}^{(\pm)} \pm \frac{1}{2} \left(1 - \frac{3\eta}{4} \right) \partial^B \left(2v_{kkB}^{(0)} - 2v_{Bkk}^{(0)} - v_{nB}^{(+)} + v_{nB}^{(-)} \right) , \tag{31}$$

$$V_A^{(\pm)} = -\partial_n v_{nA}^{(\pm)} - \partial^B v_{AB}^{(\pm)} + \partial_A v_{kk}^{(\pm)} \mp \frac{1}{2} \left(1 - \frac{3\eta}{4} \right) \partial_A \left(2v_{kkn}^{(0)} + v_{BB}^{(+)} - v_{BB}^{(-)} \right) , \tag{32}$$

where the capital indices $A, B = 1, 2$ refer to transverse directions (i.e. directions orthogonal to n), and where we sum over equal indices.

A first problem that arises in the above expressions is that with maximally dissipative boundary conditions one does not control normal derivatives ∂_n of the incoming characteristic variables at the boundary and, therefore, cannot impose the above conditions, Eqs. (31,32). However, since in Eqs. (31,32) the normal derivatives are present only on variables with speed ± 1, we can use the equations (24) to trade normal derivatives by time and transverse derivatives. Doing so one obtains

$$V_n^{(\pm)} = \pm \partial_t v_{BB}^{(\pm)} - \partial^B v_{Bn}^{(\pm)} \pm \partial^C v_{CBB}^{(0)} \pm \frac{1}{2} \left(1 - \frac{3\eta}{4} \right) \partial^B \left(2v_{kkB}^{(0)} - 2v_{Bkk}^{(0)} - v_{nB}^{(+)} + v_{nB}^{(-)} \right) , \tag{33}$$

$$V_A^{(\pm)} = \mp \partial_t v_{nA}^{(\pm)} - \partial^B v_{AB}^{(\pm)} + \partial_A v_{kk}^{(\pm)} \mp \partial^C v_{CnA}^{(0)} \mp \frac{1}{2} \left(1 - \frac{3\eta}{4} \right) \partial_A \left(2v_{kkn}^{(0)} + v_{BB}^{(+)} - v_{BB}^{(-)} \right) . \tag{34}$$

The second problem is to show well-posedness for the evolution system of the main variables. Conditions (27) together with Eqs. (33,34) do not directly translate into conditions of the form (3) for the main variables since transverse derivatives appear in the expressions (33,34) [21]. In order to get a well-posed initial-boundary problem we look for appropriate linear combinations $g = u^{(+)} - R u^{(-)}$ of in- and outgoing main variables and an appropriate coupling matrix L such that the conditions (27) with (33,34) can be incorporated in a closed set of evolution system at the boundary for the variables g and some variables with zero speed. This is discussed next.

A. Neumann boundary conditions: evolution system on each face

Consider the combinations $d_{ij} = v_{ij}^{(+)} - v_{ij}^{(-)}$ and $s_{ij} = v_{ij}^{(+)} + v_{ij}^{(-)}$ (d_{ij} stands for **d**ifference [22], and s_{ij} for **s**um). Noticing that

$$V_n^{(+)} + V_n^{(-)} = \partial_t d_{BB} - \partial^A s_{An} ,$$

$$-V_A^{(+)} + V_A^{(-)} = \partial_t s_{An} + \partial^B d_{AB} - \partial_A d_{BB} - \partial_A d_{nn} + 2\partial^C v_{CnA}^{(0)} + \left(1 - \frac{3\eta}{4} \right) \partial_A \left(2v_{BBn}^{(0)} + d_{BB} \right) ,$$

we see that one way of imposing boundary conditions such that the constraints are satisfied is through

$$0 = V_n^{(+)} + V_n^{(-)} , \tag{35}$$

$$0 = -V_A^{(+)} + V_A^{(-)} . \tag{36}$$

Equations (35,36) amount to giving as boundary conditions for the constraints' propagation a coupling between the incoming and outgoing characteristic constraint variables as in Eq. (27) with $(L_i^j) = diag(-1,1,1)$. These equations can also be seen as equations for d_{BB} and s_{nA} at the n-face, where $\hat{d}_{AB}$ and d_{nn} are a priori prescribed functions (i.e. they are not fixed by the constraints' treatment). Here $\hat{d}_{AB}$ is defined as the traceless part of d_{AB}. That is,

$$\hat{d}_{AB} = d_{AB} - \frac{1}{2}\delta_{AB}d_C{}^C$$

where $\delta_{AB} = 1$ if $A = B$ and zero otherwise.

In fact, Eqs. (35,36) do not not contain as dynamical variables only d_{BB} and s_{nA}, but also some zero speed variables. Therefore, in order to get a closed system, we need evolution equations for the zero speed variables $v_{ABn}^{(0)}$. These equations can be obtained from the evolution system of the main variables, equation (25):

$$0 = \partial_t v_{ABn}^{(0)} + \frac{1}{2}\partial_A s_{nB} . \tag{37}$$

Below we will show that equations (35,36,37) constitute a symmetrizable hyperbolic system of evolution equations at the face orthogonal to n for the variables $(d_{BB}, s_{An}, v_{ABn}^{(0)})$, where $\hat{d}_{AB} = d_{AB} - 1/2\,\delta_{AB}d_{CC}$ and d_{nn} can be freely prescribed. Since the domain we are interested in is a box, Eqs. (35,36,37) need to be evolved at each face. In order to do so one needs boundary conditions at each edge (intersection of two faces). Below we will also explain how these boundary conditions are naturally fixed by compatibility conditions.

Assuming one already has the solution to Eqs. (35,36,37) at each face, the boundary conditions for the main variables, which are of the required form (3), are the following:

$$v_{BB}^{(+)} = v_{BB}^{(-)} + d_{BB} , \qquad v_{nA}^{(+)} = -v_{nA}^{(-)} + s_{nA} , \qquad v_{nn}^{(+)} = v_{nn}^{(-)} + d_{nn} , \qquad \hat{v}_{AB}^{(+)} = \hat{v}_{AB}^{(-)} + \hat{d}_{AB} , \tag{38}$$

where d_{BB} and s_{nA} are obtained from the evolution system (35-37) at the n-face and where we can specify the gauge variable d_{nn} and the physical variable $\hat{d}_{AB}$ freely. Note that

$$d_{nn} = -2f_{nnn} , \qquad \hat{d}_{AB} = -2f_{nAB} + \delta_{AB}f_{nCC} . \tag{39}$$

In view of equation (7) and the fact that if the constraints are satisfied we have $d_{kij} = \partial_k g_{ij}$, we see that these are Neumann conditions on some components of the three-metric.

Now we go back to the evolution system (35-37) at the face and look at it in detail. In order to make the notation more compact we write $d^{(n)} = d_{BB}$, $s_A^{(n)} = s_{An}$, $h_{AB}^{(n)} = v_{ABn}^{(0)}$ for the variables associated with the face orthogonal to n. Then we have

$$\partial_t d^{(n)} = \partial^A s_A^{(n)} , \tag{40}$$

$$\partial_t s_A^{(n)} = -2\partial^B h_{BA}^{(n)} - \frac{4-3\eta}{2}\partial_A h^{(n)} - \frac{2-3\eta}{4}\partial_A d^{(n)} - \partial^B \hat{d}_{AB}^{(n)} + \partial_A d_{nn}^{(n)}, \tag{41}$$

$$\partial_t h_{AB}^{(n)} = -\frac{1}{2}\partial_A s_B^{(n)} , \tag{42}$$

where $h^{(n)}$ denotes the trace of $h_{AB}^{(n)}$. One can check that as long as $\eta \neq 2$ this system is symmetric hyperbolic with respect to the inner product associated with

$$(B,B) \equiv \left(d^{(n)}\right)^2 + 2s_A^{(n)}s_B^{(n)}\delta^{AB} + \left[(6-3\eta)\,h_{AB}^{(n)}\delta^{AB} + \frac{4-3\eta}{2}\,d^{(n)}\right]^2 + 8\hat{h}_{AB}^{(n)}\hat{h}_{CD}^{(n)}\delta^{AC}\delta^{BD} , \tag{43}$$

where $B = (d, s_A, h_{AB})$ and $\hat{h}_{AB}$ denotes the trace-less part of h_{AB}, $\hat{h}_{AB} = h_{AB} - (\delta_{AB}h_C{}^C)/2$.

B. Neumann boundary conditions: Compatibility conditions at the edges of the faces

The system we just introduced is defined on each of the six n-faces. The faces themselves have boundaries: the edges that join them. We need to ensure that the system on each face is well posed taking into account the boundary conditions at the edges. Since each edge is shared by two faces, this will translate into compatibility conditions among the various systems.

The system of interest has, at each edge, two ingoing modes with speeds $\sqrt{3/2}$ and 1 and two outgoing modes with speeds $-\sqrt{3/2}$ and -1. At face n, the corresponding characteristic variables with respect to a direction m^A, are

$$w_{\pm\sqrt{3/2}} = \pm\frac{1}{2}\sqrt{\frac{2}{3}}\,s_m^{(n)} - \frac{1}{3}\left[2h_{mm}^{(n)} + \frac{4-3\eta}{2}h^{(n)} + \frac{2-3\eta}{4}d^{(n)}\right],$$
$$w_{\pm 1} = \pm s_p^{(n)} - 2h_{mp}^{(n)},$$

where p is a transverse direction orthogonal to m (and n).

In terms of the original variables, the variables $d^{(n)}, s_A^{(n)}, h_{AB}^{(n)}, d_{nn}^{(n)}, \hat{d}_{AB}^{(n)}$ on the n-face are

$$d^{(n)} = -2f_{nBB}, \qquad s_A^{(n)} = 2K_{nA}, \qquad h_{AB}^{(n)} = f_{ABn}, \qquad d_{nn}^{(n)} = -2f_{nnn}, \qquad \hat{d}_{AB}^{(n)} = -2f_{nAB} + \delta_{AB}f_{nCC}.$$

Now let $\{n, m, p\}$ be orthogonal directions naturally associated with the box (say, the standard Cartesian coordinates $\{x, y, z\}$). If we compare these variables in two different directions n and m, say, we obtain the following compatibility conditions:

$$s_m^{(n)} = s_n^{(m)}, \tag{44}$$
$$-2h_{mp}^{(n)} = \hat{d}_{pn}^{(m)}, \tag{45}$$
$$h_{pm}^{(n)} = h_{pn}^{(m)}. \tag{46}$$

On the other hand, we have, by definition of the characteristic variables,

$$w_{+\sqrt{3/2}}^{(n)} - w_{-\sqrt{3/2}}^{(n)} = \sqrt{\frac{2}{3}}\,s_m^{(n)}, \qquad w_{+1}^{(n)} + w_{-1}^{(n)} = -4h_{mp}^{(n)}.$$

Therefore, the correct boundary conditions at the edges are

$$w_{+\sqrt{3/2}}^{(n)} = w_{-\sqrt{3/2}}^{(n)} + \sqrt{\frac{2}{3}}\,s_n^{(m)}, \tag{47}$$
$$w_{+1}^{(n)} = -w_{-1}^{(n)} + 2\hat{d}_{np}^{(m)}. \tag{48}$$

Notice that up to now the quantities that were freely specified were d_{nn} and $\hat{d}_{AB}$ one each of the n-faces. Therefore, in order to fix the boundary data for the systems that are defined on these faces we also have to a priori specify the quantities $s_n^{(m)} = 2K_{nm} = s_m^{(n)}$ at the edges defined by the intersection of faces n and m.

Imposing these boundary conditions automatically implies that the compatibility conditions (44) and (45) are satisfied. On the other hand, we have at the edge joining the faces n and m,

$$\partial_t\left(h_{pm}^{(n)} - h_{pn}^{(m)}\right) = -\frac{1}{2}\partial_p\left(s_m^{(n)} - s_n^{(m)}\right) = 0$$

where the last equality follows from imposing the boundary conditions at the edges. Therefore, these boundary conditions also imply that the compatibility condition (46) is satisfied through evolution provided it does so initially.

C. Dirichlet boundary conditions

In a similar way to the Neumann case, one can obtain a closed system at the boundary for the variables $(s_{BB}, d_{An}, v_{Ann}^{(0)}, v_{ABB}^{(0)}, v_{BBA}^{(0)})$ by requiring

$$0 = V_n^{(+)} - V_n^{(-)}, \tag{49}$$
$$0 = V_A^{(+)} + V_A^{(-)}, \tag{50}$$

where

$$V_n^{(+)} - V_n^{(-)} = \partial_t s_{BB} - \partial^A d_{An} + 2\partial^A v_{ABB}^{(0)} + \left(1 - \frac{3\eta}{4}\right)\partial_A\left(2v_{BBA}^{(0)} - 2v_{Akk}^{(0)} - d_{nA}\right),$$
$$-V_A^{(+)} - V_A^{(-)} = \partial_t d_{An} + \partial^B s_{AB} - \partial_A s_{BB} - \partial_A s_{nn}.$$

One also has to take into account the evolution equations for the zero speed variables:

$$0 = \partial_t v_{Aij}^{(0)} + \frac{1}{2} \partial_A s_{ij} . \tag{51}$$

In this case, one can freely specify s_{nn} and $\hat{s}_{AB}$, which corresponds to Dirichlet conditions on some components of the extrinsic curvature:

$$s_{nn} = 2K_{nn} \quad , \quad \hat{s}_{AB} = 2K_{AB} - \delta_{AB} K_{CC} . \tag{52}$$

Assuming one already has the solution to Eqs. (49,50) at each face, the boundary conditions for the main variables, which are of the required form (3), are the following:

$$v_{BB}^{(+)} = -v_{BB}^{(-)} + s_{BB} , \qquad v_{An}^{(+)} = v_{An}^{(-)} + d_{An} , \qquad v_{nn}^{(+)} = -v_{nn}^{(-)} + s_{nn} , \qquad \hat{v}_{AB}^{(+)} = -\hat{v}_{AB}^{(-)} + \hat{s}_{AB} . \tag{53}$$

In terms of the variables $s = s_{BB}$, $d_A = d_{An}$, $h_A = 3\eta\, v_{ABB}^{(0)} + (4 - 3\eta)(v_{BBA}^{(0)} - v_{Ann}^{(0)})$, we can rewrite the boundary system (49-51) as

$$\partial_t s = \frac{8 - 3\eta}{4} \partial^A d_A - \frac{1}{2} \partial^A h_A , \tag{54}$$

$$\partial_t d_A = \frac{1}{2} \partial_A s + \partial_A s_{nn} - \partial^B \hat{s}_{BA} , \tag{55}$$

$$\partial_t h_A = -\frac{4 + 3\eta}{4} \partial_A s + \frac{4 - 3\eta}{2} \left(\partial_A s_{nn} - \partial^B \hat{s}_{BA} \right) . \tag{56}$$

This system is symmetrizable hyperbolic with respect to the inner product associated with

$$(B, B) = 4s^2 + 8d^A d_A + \left(h^A - \frac{4 - 3\eta}{2} d^A \right) \left(h_A - \frac{4 - 3\eta}{2} d_A \right) \tag{57}$$

where $B = (s, d_A, h_A)$.

There is one ingoing and one outgoing mode, with speeds $\pm\sqrt{3/2}$, respectively. The corresponding variables in a direction m^A are

$$w_{\pm\sqrt{3/2}} = \sqrt{3/2}\, s \pm \frac{1}{2} \left(\frac{8 - 3\eta}{2} d_m - h_m \right) .$$

We now turn to the compatibility at the edges of the faces. In terms of the original variables one has

$$s^{(n)} = 2K_{BB} , \qquad d_A^{(n)} = -2f_{nnA} , \qquad h_A^{(n)} = 3\eta\, f_{ABB} + (4 - 3\eta)(f_{BBA} - f_{Ann}),$$
$$s_{nn}^{(n)} = 2K_{nn} , \qquad \hat{s}_{AB}^{(n)} = 2K_{AB} - \delta_{AB} K_{CC} .$$

From these expressions one can see that the only compatibility conditions are

$$s^{(n)} = 2 \left(s_{mm}^{(m)} - \hat{s}_{mm}^{(n)} \right) , \tag{58}$$

$$s_{mm}^{(m)} + 2\hat{s}_{nn}^{(m)} = s_{nn}^{(n)} + 2\hat{s}_{mm}^{(n)} . \tag{59}$$

Equation (58) fixes the boundary data for the ingoing variable:

$$w_{+\sqrt{3/2}}^{(n)} = -w_{-\sqrt{3/2}}^{(n)} + 4\sqrt{3/2}(s_{nn}^{(m)} - \hat{s}_{mm}^{(n)}). \tag{60}$$

This condition will be used in the energy estimate (63) in order to prove well-posedness for the system at each face in the Dirichlet case. On the other hand, Eq. (59) is a compatibility condition at the intersection of faces n and m for the free boundary data.

VI. WELL POSEDNESS

We start by showing that the conditions (35,36) and (49,50), for the Neumann and Dirichlet case, respectively, imply that the evolution system for the constraints is well-posed. Then we derive energy estimates for the closed system of evolution equations at each face, and using these estimates we show that the evolution of the main system is well-posed as well.

Since we have already shown that all the evolution equations involved in our constraint-preserving treatment are symmetric hyperbolic, and since we have already cast all boundary conditions in maximally dissipative form, the main purpose of this section is to explicitly show that the different couplings are "small enough" with respect to the corresponding symmetrizers, in the sense discussed in Section II.

A. Constraint propagation

Here we derive an estimate for the growth of the energy

$$E_{constraints} = \int_\Omega (U, U)\, d^3x,$$

where (U, U) is defined in (20). Taking a time derivative and using equations (13-19) we obtain, after integrations by parts,

$$\frac{d}{dt} E_{constraints} = 2 \int_{\partial\Omega} \left[\frac{4 - 2\eta}{\eta} CC_n - C^i S_{ni} - C^i \tilde{A}_{ni} \right] d\sigma,$$

where n^k is the unit outward normal to the boundary $\partial\Omega$ of the domain. Expressing the integrand in terms of characteristic variables defined in (26), we obtain

$$\frac{d}{dt} E_{constraints} = \frac{1}{2} \int_{\partial\Omega} \delta^{ij} \left(V_i^{(+)} V_j^{(+)} - V_i^{(-)} V_j^{(-)} \right) d\sigma = 0,$$

where the last equation follows from the conditions (35,36) and (49,50) in the Neumann and Dirichlet cases, respectively. Therefore, the initial-boundary value problem for the constraints is well-posed. In particular, this implies that zero initial data for the constraints implies that the constraints are zero at later times as well.

B. Face systems

1. Neumann conditions

In order to show well-posedness for each system defined on the n-face Γ_n we consider the corresponding energy norm for the Neumann case,

$$E_{(\Gamma_n, N)} = \int_{\Gamma_n} (B, B)\, d\sigma,$$

where (B, B) is defined in (43). Taking a time derivative and using the above evolution equations we obtain

$$\frac{d}{dt} E_{(\Gamma_n, N)} = \int_{\partial\Gamma_n} \left[6\sqrt{\frac{3}{2}} \left(w^2_{+\sqrt{3/2}} - w^2_{-\sqrt{3/2}} \right) + 4 \left(w^2_{+1} - w^2_{-1} \right) \right] ds + 4 \int_{\Gamma_n} s^A \left(\partial_A d_{nn} - \partial^B \hat{d}_{AB} \right) d\sigma. \tag{61}$$

We use the boundary conditions (47,48), with $s_n^{(m)} = 0$ and $\hat{d}_{np}^{(m)} = 0$ for the moment, to get rid of the first term on the RHS of Eq. (61). Using Schwarz's inequality, the second term is estimated as follows:

$$4 \int_{\Gamma_n} s^A \left(\partial_A d_{nn} - \partial^B \hat{d}_{AB} \right) d\sigma \le 4\, f\, E_{(\Gamma_n, N)}^{1/2},$$

where

$$f^2 = \int_{\Gamma_n} \left(\partial_A d_{nn} - \partial^B \hat{d}_{AB} \right) \left(\partial^A d_{nn} - \partial_C \hat{d}^{AC} \right) d\sigma.$$

Therefore, we end up with the estimate

$$E_{(\Gamma_n, N)}(t)^{1/2} \le E_{(\Gamma_n, N)}(0)^{1/2} + 2 \int_0^t f(s)\, ds \tag{62}$$

and see that the energy is bounded and the bound is determined by the norm f of the free data on the boundary.

The assumptions $s_n^{(m)} = 0 = \hat{d}_{np}^{(m)}$ can be easily relaxed with an argument similar to the one we presented in the paragraph following equation (3). That is, one defines a new variable that incorporates the inhomogeneity in the energy generated by the term that appears in the case of a non-smooth boundary and obtains an energy estimate for the new variable, since the variable redefinition is finite and well defined. Well posedness follows immediately.

2. Dirichlet conditions

The Dirichlet case is similar to the Neumann one. The energy is now given by

$$E_{(\Gamma_n,D)} = \int_{\Gamma_n} (B,B)\, d\sigma,$$

with (B,B) given by Eq. (57). We obtain the estimate

$$\frac{d}{dt} E_{(\Gamma_n,D)} \leq \sqrt{8/3} \int_{\partial\Gamma_n} \left(w^2_{+\sqrt{3/2}} - w^2_{-\sqrt{3/2}} \right) ds + 2\, \tilde{f} E^{1/2}_{(\Gamma_n,D)}, \tag{63}$$

where

$$\tilde{f}^2 = \int_{\Gamma_n} \left(\partial_A s_{nn} - \partial^B \hat{s}_{AB} \right) \left(\partial^A s_{nn} - \partial_C \hat{s}^{AC} \right) d\sigma,$$

and we can proceed as in the Neumann case, using in this case the boundary condition (60).

C.   Main system

Having obtained a bound for each closed system defined on a face we can obtain a bound for the main evolution variables by the standard techniques described in section II where we now have the boundary conditions (38) and (53) for the Neumann and Dirichlet cases, respectively, which are of the required form (3).

VII.   SUMMARY

The well posed, constraint preserving boundary conditions presented in this paper can be summarized as follows.

A.   Neumann case

- Free data:

  At each face (say, the n-one), the three quantities d_{nn} and $\hat{d}_{AB}$ must be a priori defined (subject to standard compatibility conditions with the initial data). These quantities correspond to the normal (that is, in the n direction) derivative of the normal and the transverse, traceless parts of the three metric, see Eq. (39). One of these variables is gauge and the other two are physical, in the sense discussed in Section IV. In addition, the quantities $s^{(n)}_m = 2K_{nm}$ should be prescribed at each edge defined by the intersection of faces n and m. These quantities must satisfy the compatibility conditions $s^{(m)}_n = s^{(n)}_m$.

- Evolution systems on faces:

  The $2D$ symmetric hyperbolic 7×7 system (40,41,42) is evolved on each face. This system needs boundary conditions at the edges of the corresponding face. They are given by equations (47, 48).

  The solution to each of these systems provides the three quantities $d^{(n)}_{BB}$ and $s^{(n)}_{nA}$ at each of the six n-faces.

- Main evolution system:

  The main system (21,22) is evolved in the $3D$ domain. This system needs boundary conditions, at each face, for the six incoming characteristic modes. These boundary conditions at, say face n, are given by Eq. (38), where the needed information for three of these boundary conditions is provided by the a priori specified d_{nn} and $\hat{d}_{AB}$, while the other three are given by $d^{(n)}_{BB}$ and $s^{(n)}_{nA}$.

B. Dirichlet case

- Free data:

  At each face (say, the n one), now the three quantities s_{nn} and $\hat{s}_{AB}$ must be a priori given. They correspond to the time derivative of the normal and transversal, traceless part of the three metric, see Eq. (52). These three quantities have to satisfy the standard compatibility conditions with the initial data, but also some compatibility conditions at edges, Eqs. (58,59). They are also gauge and physical variables, in the sense discussed in Section IV.

- Evolution systems on faces:

  The $2D$ symmetric hyperbolic 5×5 system (54,55,56) is evolved on each face. This system needs boundary conditions at each edge. They are given by Eq. (60).

  The solution to each of these systems provides the three quantities $s_{BB}^{(n)}$ and $d_{An}^{(n)}$ at each of the six n-faces.

- Main evolution system:

  The main system (21,22) is evolved in the $3D$ domain. This system needs boundary conditions, at each face, for the six incoming characteristic modes. These boundary conditions are given by (53), where the needed information in three of these boundary conditions is provided by the a priori specified s_{nn} and $\hat{s}_{AB}$, while the other three are given by $s_{BB}^{(n)}$ and $d_{An}^{(n)}$.

VIII. CONCLUSIONS:

We have studied the system of evolution equations for the constraints for a subfamily of the generalized Einstein-Christoffel symmetric hyperbolic system. We have shown how to give boundary data for the constraints in such a way that it translates into boundary data for the main system that yields a well posed problem, both for the main system and the system of evolution equations for the constraints. We have studied the case of a boundary that is not smooth, as is the case of the usual cubic boxes used in numerical relativity. This required additional care at the boundaries of each face, with the ensuing compatibility conditions. It should be noted that the energy estimates derived do not necessarily guarantee the existence of a smooth solution in the presence of corners even with the compatibility conditions we presented. Further work is needed to establish smoothness of the solution.

Our analysis was carried out for the case of linearized gravity around Minkowski space-time. It is expected that similar techniques will be useful in the case of other background space-times and also in the non-linear case. We will discuss these generalizations in future papers. Also, since we have followed a systematic approach and have not taken any advantage of the gauge choice, in principle it should be possible to apply the same procedure to symmetric hyperbolic formulations with live gauges [16].

We have also found that, at least with the formulation of Einstein's equations here used (the generalized EC), the Neumann and Dirichlet cases are in fact the *only* ones for which well posedness can be established through the techniques used in this paper (see the appendix). More specifically, we have found that these two cases are the only ones in which closed systems at the faces can be obtained. However, this does not mean that these are the only well posed cases, since in the initial-boundary value problem an energy estimate is a sufficient but not necessary condition for well posedness. Also, giving the appropriate $(d_{nn}, d^{\hat{(n)}}{}_{AB}, s_{nm})$ data in the Neumann case, or the appropriate $(s_{nn}, \hat{s}_{AB})$ in the Dirichlet case, one should be able to recover any spacetime in any slicing. This is because our constraint-preserving treatment makes sure that one is solving not only Einstein's evolution equations but also the constraints. But it does not make any restriction on the space of solutions to the Einstein equations. However, it is not clear how to choose these "appropriate" boundary conditions, without any a priori knowledge of the solutions, in order to model an isolated source, given that the boundaries are at a finite distance. This same problem appears in similar approaches [6, 9]. One possible solution is to provide these functions through Cauchy-characteristic [9, 17] or Cauchy-perturbative matching [18], or to resort arguments using the peeling property. Another possibility would be to impose a "no incoming radiation" condition. However, it is not clear how to do this within formulations that have as extra variables first but not second spatial derivatives of the three metric. This might be remedied by repeating this construction for other systems of evolution equations where the variables that represent gravitational radiation at the boundary play a more central role, as in [3]. For instance, a system of evolution equations of higher order where the Weyl tensor is the fundamental variable, would be suitable for this purpose.

With the results of this paper one can now assure that both the evolution equation for the main variables of the problem and the evolution equations for the constraints are well posed on a manifold with (non smooth) boundary. This allows to evolve initial data that satisfy the constraints beyond their domain of dependence, as is of interest in

numerical simulations of the binary black hole problem. Moreover, well posedness opens the possibility of constructing numerical schemes for which numerical stability can be rigorously proved.

APPENDIX A: CLOSING THE BOUNDARY SYSTEM

In Section V we showed how to construct a closed system at the boundary by taking appropriate linear combinations of characteristic variables. We also chose a particular combination of ingoing and outgoing constraints and showed that it gives rise to a system of partial differential equations that lives on the boundary. To close this system we had to include the evolution of some zero speed modes. A question that arises is whether there other ways of closing the boundary system, apart from the Neumann (40-42) and the Dirichlet (54-56) case.

To answer this question we will make no assumptions on the coupling matrices R and L. The boundary condition for the system of the constraints is assumed to be

$$V_i^{(+)} = L_i{}^j V_j^{(-)} \,, \tag{A1}$$

where $L_i{}^j$ is a 3×3 coupling matrix and $V_i^{(\pm)}$ is given in (33) and (34). At the boundary data must be given to the ingoing modes. We will assume that they satisfy

$$v_{ij}^{(+)} = R_{ij}{}^{kl} v_{kl}^{(-)} + b_{ij} \,, \tag{A2}$$

where $R_{ij}{}^{kl}$ is a 6×6 coupling matrix and b_{ij} is the boundary data. If we insert (A2) into (A1) we obtain a system which contains derivatives of the boundary data b_{ij}, of the outgoing modes $v_{ij}^{(-)}$, and of the zero speed modes $v_{Tij}^{(0)}$. This system can be solved for the time derivatives of three of the b_{ij}, namely b_{BB} and b_{nA}. To the remaing b_{ij} one can give arbitrary data and consider them as source terms. In order to close the system the coefficients that multiply terms containing outgoing modes $v_{ij}^{(-)}$ must vanish. After imposing this condition the coupling matrices R and L, which in general depend on 45 parameters, depend on one parameter only (apart from η). The zero speed modes that appear in the rhs of this system cannot be eliminated by any choice of this parameter. Therefore in order to close the system one has to enlarge it by including the evolution of at least the zero speed modes that appear in the rhs. The requirement that the evolution of these zero speed modes do not contain any spatial derivatives of outgoing modes, forces the couplings R and L to be the ones the we used in subsections (V A) and (V C).

Summarizing, the Neumann and the Dirichlet cases are the only ways that one can obtain a closed system at the boundary. Furthermore, as we have shown in this paper, the boundary system is symmetric hyperbolic and the coupling matrices R and L are "not too large". Any other choice of coupling matrices would lead to a system for which the techniques used in this paper to prove well-posedness cannot be applied.

[1] P. D. Lax, R. S. Phillips, Commun. Pure Appl. Math. **13**, 427 (1960).
[2] G. Calabrese, J. Pullin, O. Sarbach and M. Tiglio, Phys. Rev. D **66**, 041501 (2002)
[3] H. Friedrich and G. Nagy, Comm. Math. Phys. **201**, 619 (1999).
[4] J. M. Stewart, Class. Quantum Grav. **15**, 2865 (1998).
[5] M. S. Iriondo and O.A. Reula, Phys. Rev. D **65**, 044024 (2002).
[6] B.Szilagyi, B. Schmidt, and J. Winicour, Phys. Rev. D **65**, 064015 (2002).
[7] J. M. Bardeen and L. T. Buchman, Phys. Rev. D **65**, 064037 (2002).
[8] G. Calabrese, L. Lehner, and M. Tiglio, Phys. Rev. D **65**, 104031 (2002).
[9] B. Szilagyi and J. Winicour, *"Well-Posed Initial-Boundary Evolution in General Relativity,"* arXiv:gr-qc/0205044.
[10] L.E. Kidder, M.A. Scheel, and S.A. Teukolsky, Phys. Rev. D **64**, 064017 (2001).
[11] A. Anderson and J.W. York, Jr., Phys. Rev. Lett. **82**, 4384 (1999).
[12] H.O. Kreiss, J. Lorenz, *"Initial-Boundary Value Problems and the Navier-Stokes Equations,"* Academic Press, (1989).
[13] P. Secchi, Diff. Int. Eq. **9**, 671 (1996); Arch. Rat. Mech. Anal. **134**, 595 (1996).

[14] J. Rauch, Trans. Am. Math. Soc. **291**, 167 (1985).

[15] L. Lindblom, M. Scheel, *"Energy Norms and the Stability of the Einstein Evolution Equations,"* arXiv:gr-qc/0206035

[16] O. Sarbach and M. Tiglio, *"Exploiting gauge and constraint freedom in hyperbolic formulations of Einstein's equations,* to appear in Phys. Rev. **D**, arXiv:gr-qc/0205086.

[17] J. Winicour, *Characteristic evolution and matching*, Living Reviews in Relativity, arXiv:gr-qc/0102085.

[18] L. Rezzolla *et al.*, Phys. Rev. D **59**, 064001 (1999); M. E. Rupright *et al.*, Phys. Rev. D **58**, 044005 (1998); A. Abrahams *et al.*, Phys. Rev. Lett. **80**, 1812 (1998).

[19] In the sense that precisely $(u, R^T H R u) \leq (u, H u) \ \forall u$.

[20] This system corresponds to System 3 of [10] with their parameter $\hat{z}$ set to zero.

[21] Note that this difficulty does not arise in $1+1$ dimensions since then there are no transverse derivatives. In that case one can simply set the ingoing constraint violating variables $v_{BB}^{(+)}$ and $v_{nA}^{(+)}$ to zero.

[22] Notice that d is used for this quantity as well as for $d_{kij} = \partial_k g_{ij}$, equation (7). There does not seem to be risk of confusion since both objects have different number of indexes.